PHYSICAL REVIEW E

STATISTICAL PHYSICS, PLASMAS, FLUIDS,
AND RELATED INTERDISCIPLINARY TOPICS

THIRD SERIES, VOLUME 48, NUMBER 5 NOVEMBER 1993

RAPID COMMUNICATIONS

The Rapid Communications section is intended for the accelerated publication of important new results. Since manuscripts submitted to this section are given priority treatment both in the editorial office and in production, authors should explain in their submittal letter why the work justifies this special handling. A Rapid Communication should be no longer than 4 printed pages and must be accompanied by an abstract. Page proofs are sent to authors.

Saddle-point bifurcation and onset of large-scale stochasticity in 1.5-degree-of-freedom Hamiltonian systems

Sadruddin Benkadda and Yves Elskens
Equipe Turbulence Plasma, Université de Provence, Institut Méditerranéen de Technologie, Technopôle de Château-Gombert, F-13451 Marseille Cedex 20, France
(Received 23 July 1993)

A recent numerical work [P. Hennequin, M. A. Dubois, and R. Nakach, Phys. Lett. A **164**, 259 (1992)] analyzed the effect of the separatrix shape on stochasticity onset in a two-wave Hamiltonian system and stated in particular a very unexpected result: it linked a lowering of the stochasticity threshold to the flattening of a "separatrix angle" and to the bifurcation of the associated saddle point. In this paper, we show that there is no simple dependence of this threshold on a given control parameter of the separatrix shape.

PACS number(s): 05.45.+b, 03.20.+i

I. INTRODUCTION

Recently, in a numerical work the effect of the separatrix shape on the stochasticity onset in a two-wave Hamiltonian system was analyzed. In particular, the dependence of the stochasticity threshold on the "separatrix angle" at the hyperbolic fixed point was studied [1]. The conclusion of that work was that the flattening of the angle has a very significant effect on the reduction of the stochasticity threshold, i.e., that a change in the nature of fixed points or simple periodic orbits significantly affects chaotic transport. Using this result, the authors proposed a scenario for the unsolved problem of internal disruptions in tokamaks [2], wherein they related the catastrophic behavior observed during those disruptions to the flattening of the separatrix angle at the hyperbolic fixed point.

In the present paper we use the same Hamiltonian introduced by the previous authors as an example to demonstrate that (i) the stochasticity threshold is independent of the value of the separatrix angle at the hyperbolic point, and (ii) there is no simple dependence of this threshold on any given parameter controlling the separatrix shape, such as the flatness of this separatrix at the elliptic fixed point or the number of its ripples.

II. STUDY OF THE MODEL HAMILTONIAN FAMILY

Following Ref. [1], we consider the Hamiltonian

$$H(x,v,t)=\frac{v^2}{2}+E_1V(x+t)+E_2V(x-t) \qquad (1)$$

with potential

$$V(q)=\cos\left[q-\frac{\alpha}{m}\sin(mq)\right] \qquad (2)$$

and $(x,v,t)\in S_1\times\mathbb{R}\times S_1$ (S_1 is the circle). This four-parameter family of models describes for instance the motion of a charged particle in the potential of two electrostatic waves, traveling at velocities ± 1; the wave amplitudes are $E_1\geq 0$ and $E_2\geq 0$; the parameters $\alpha\in\mathbb{R}$ and $m\in\mathbb{N}_0$ control the shape of the waves. Figure 1(a) displays the potential V for some values of α and m; for

$\alpha=1$, the unstable equilibrium at $q=0$ is degenerate. Up to a linear change of variables, the model [(1) and (2)] include the two-wave paradigm Hamiltonian of Escande and Doveil [3] in two ways: (i) for $\alpha=0$, regardless of m; (ii) in the limit $m\to\infty$, regardless of α (however, the vector field, i.e., the dynamics, generated by H is not defined in this limit).

A. One-wave Hamiltonian

For $E_1=0$ ($E_2=0$), the Hamiltonian (1) is integrable and describes the motion of a charged particle in the potential of a single modulated wave with velocity $+1$ (-1). A Galilean transformation reduces (1) and (2) to

$$H_1(q,p,t)=\frac{p^2}{2}+E\cos\left[q-\frac{\alpha}{m}\sin(mq)\right]. \quad (3)$$

For $\alpha\geq 0$, it admits at least two fixed points; their stability does not depend on m:

(i) $(\pi,0)$ has eigenvalues $\pm i(1+\alpha)\sqrt{E}$; it is elliptic.

(ii) (0,0) has eigenvalues $\pm|1-\alpha|\sqrt{E}$; it is hyperbolic for $\alpha\neq 1$ and parabolic for $\alpha=1$; at $\alpha=1$, it undergoes a degenerate bifurcation giving rise to two elliptic points $(\pm(1/m)\sqrt{2(1-1/\alpha)}+\cdots,0)$ and two saddle points $(\pm(1/m)\sqrt{6(1-1/\alpha)}+\cdots,0)$.

Regardless of the values of α and m, the single-wave Hamiltonian shows that the particle is trapped for energy values $-E\leq H<E$ and untrapped (circulating) for energy values $H>E$; at separatrix energy $H=E$, its maximal velocity is $2\sqrt{E}$. The phase portrait of H_1 [see Fig. 1(b) for $\alpha=1$, $m=3$] shows a single (or triple if $\alpha>1$) resonance with a half width of $2\sqrt{E}$. The parameter α controls the angle χ between the stable and unstable directions at the hyperbolic point ($\tan\chi=|1-\alpha|$) and m controls the ripples of the potential and the separatrix.

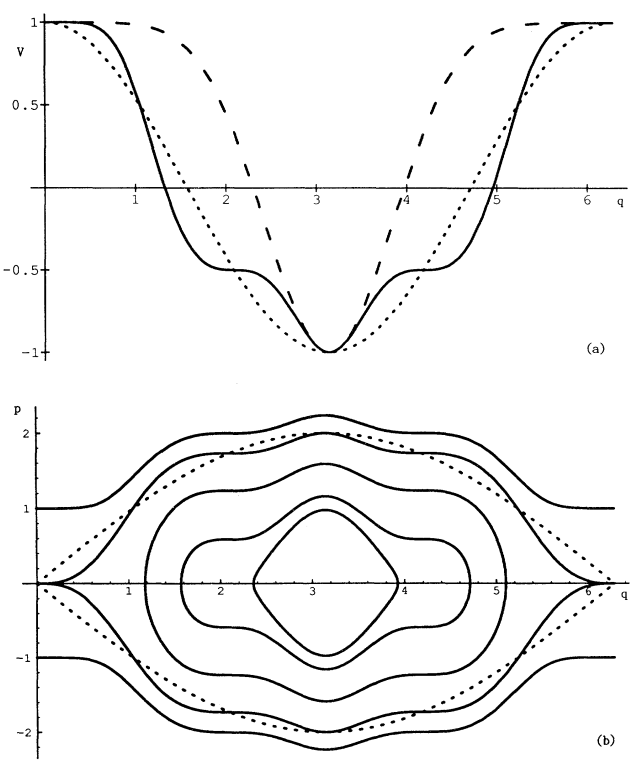

FIG. 1. (a) Potential for $\alpha=0$ (dots); $\alpha=1$, $m=1$ (dashes); and $\alpha=1$, $m=3$ (solid line). (b) Phase portrait of the integrable single-wave Hamiltonian for $\alpha=1$, $m=3$, $E=1$. Dotted lines show the separatrix for $\alpha=0$.

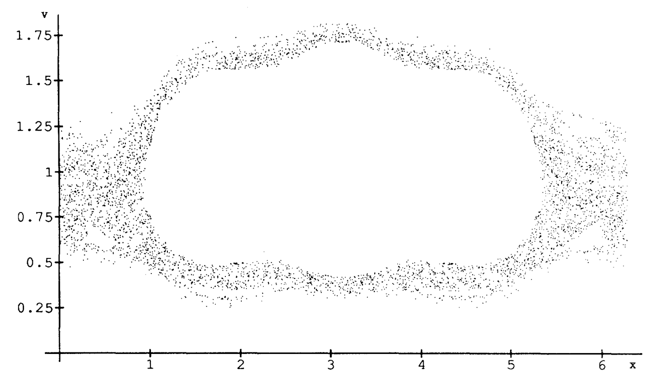

FIG. 2. 5000 points of a trajectory for $m=3$, $\alpha=1$, $s=0.72$.

The fixed points of H_1 correspond to periodic trajectories of H. However, the phase portrait of H_1 also gives the stroboscopic Poincaré section of the flow of H at times $t=0\ (\mathrm{mod}2\pi)$, with $p=v-1$, $q=x-t$.

B. Two-wave phase portrait

For $E_1>0$ and $E_2>0$, the model (1) is not integrable. We discuss only the case $E_1=E_2=E>0$ and define the resonance overlap parameter $s=2\sqrt{E}$. For $0<s\ll 1$, the phase portrait of H shows stochastic domains separated by KAM tori; the main stochastic regions appear in the vicinity of the homoclinic trellises originating from the separatrixes of the two main resonances, associated with the hyperbolic periodic orbits $(0,\pm 1+O(s))$. For increasing s, the model undergoes a transition to large-scale chaos at a value $s^*(\alpha,m)$ for which the two main stochastic seas merge. Figure 2 displays one trajectory for $m=3$, $\alpha=1$, $s=0.72<s^*(1.0,3)$ to illustrate the effect of the potential modulation on the shape of invariant islands and tori.

The symmetries of the model [(1) and (2)] for $E_1=E_2$ provide us with a simple criterion to estimate s^*. Since the Hamiltonian is even with respect to each phase-space variable x, v, and t and since the Poincaré section is performed at $t=0$, we note that if $v=\psi(x)$ is the equation of an invariant torus in this section, then $v=-\psi(x)$ is the equation of a symmetric torus. Thus the boundary of the upper main stochastic sea intersects the axis $v=0$ if and only if $s>s^*(\alpha,m)$.

We estimate s^* numerically as follows. Given α, m, and s, we choose 40 initial data at random in a rectangle $[0.0,0.3]\times[0.7,0.9]$ at $t=0$; we compute their trajectories (using a leapfrog integrator with 400 steps per Poincaré period to allow for $m\leq 10$) over 10^4 Poincaré periods; if at least one trajectory reaches a value $v<0$ in

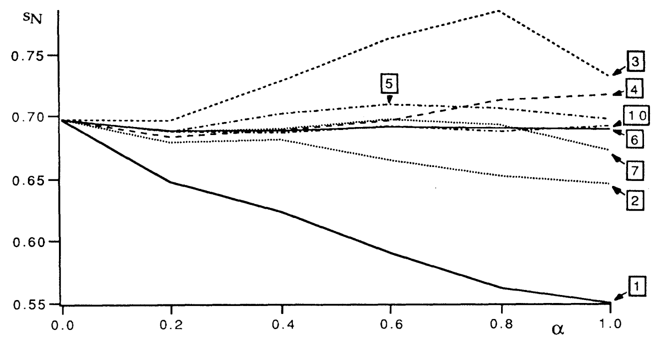

FIG. 3. Numerical estimate of large-scale stochasticity threshold s_N vs α for $1 \le m \le 7$ and $m = 10$.

the Poincaré section, then $s > s^*(\alpha, m)$.

Figure 3 displays the estimate threshold $s_N(\alpha, m)$, such that no trajectory reaches $v < 0$ if $s \le s_N(\alpha, m) - 10^{-2}$, and at least one does for $s = s_N(\alpha, m)$. All lines start at $\alpha = 0$ from the paradigm Hamiltonian's value $s^* = 0.69$, but they behave quite differently for increasing α:

(i) For $m = 1$, in agreement with observations in Ref. [1], s_N decreases monotonically to 0.55 at $\alpha = 1$.

(ii) For $m = 2$, s_N decreases only to 0.64.

(iii) For $m = 3$, s_N *increases* to 0.78 at $\alpha = 0.8$ and is still $0.73 > 0.69$ at $\alpha = 1$.

(iv) For $m \ge 4$, s_N remains close to 0.69 (as one would expect for $m \to \infty$), but shows no simple dependence on (α, m).

Thus neither α nor m controls the large-scale stochasticity threshold $s^*(\alpha, m)$ in a straightforward way.

We can also estimate analytically the large-scale stochasticity threshold using the renormalization technique of Ref. [4]. By definition,

$$s^*(\alpha, m) = \sup_{-1 < \omega < 1} s_A(\omega; \alpha, m) , \quad (4)$$

where $s_A(\omega; \alpha, m)$ is the value of s for which the invariant torus T_ω with average phase velocity ω gets broken. To estimate $s_A(\omega; \alpha, m)$, one performs a Kolmogorov change of variables to rewrite the dynamics generated by $H(x, v, t)$ in the vicinity of T_ω in the form of the dynamics generated by the paradigm two-wave Hamiltonian plus a "small" perturbation:

$$H(\varphi, I, t) = \frac{I^2}{2} - M \cos\varphi - P \cos(\varphi - t) , \quad (5)$$

where M and P depend on (ω, α, m, s). In particular, for $\omega = (\sqrt{5} - 1)/2$, $\alpha = 1$, $m = 1$, the large-scale stochasticity threshold is estimated as $s_A = 0.59$, in reasonable agreement with numerical observation.

III. CONCLUSION AND COMMENTS

It would have been nice to relate, even roughly, the large-scale stochasticity threshold in two-wave models, measured by a critical resonance overlap parameter s^* or by a critical wave amplitude $E^* = s^{*2}/4$, to a "single-wave" quantity, such as the flatness parameter α at a hyperbolic point. Such a relation would save on long numerical experiments. But, such a connection would be surprising: the transition to large-scale stochasticity as s increases is a *global* bifurcation, involving the destruction of a whole torus in phase space (a curve in the Poincaré section), whereas the flatness effect involves only a *local* bifurcation, at a fixed point in the Poincaré section. Moreover, one does not expect even locally strong chaos to occur near a parabolic point (whose Liapunov exponents vanish): why then should it favor large-scale chaos?

To summarize, we have shown that the large-scale stochasticity threshold in a two-wave Hamiltonian system has no simple dependence on a given control parameter of the separatrix shape. Therefore any physical theory relying on such a relationship would be doubtful.

ACKNOWLEDGMENTS

The analytical estimate was kindly supplied by Ph. Gabbai. The authors acknowledge interesting discussions with M. A. Dubois, P. Hennequin, R. Nakach, and R. Sabot. One of the authors (S.B.) was partly supported by the Commissariat à l'Energie Atomique (DRFC Cadarache). The Equipe Turbulence Plasma is a team of "Laboratoire de Physique des Interactions Ioniques et Moléculaires, Unité de Recherche Associeé No. 773 au CNRS."

[1] P. Hennequin, M. A. Dubois, and R. Nakach, Phys. Lett. A **164**, 259 (1992).

[2] M. A. Dubois, P. Hennequin, R. Nakach, and R. Sabot, in Proceedings of the International Conference on Plasma Physics, Innsbrück, edited by W. Freysinger, K. Lackner, R. Schrittwieser, and W. Lindinge [Europhys. Conf. Abstr. **16C**, Part II, 1409 (1992).

[3] D. F. Escande and F. Doveil, Stat. Phys. **26**, 257 (1981).

[4] D. F. Escande, M. S. Mohamed-Benkadda, and F. Doveil Phys. Lett. A **101**, 309 (1984).